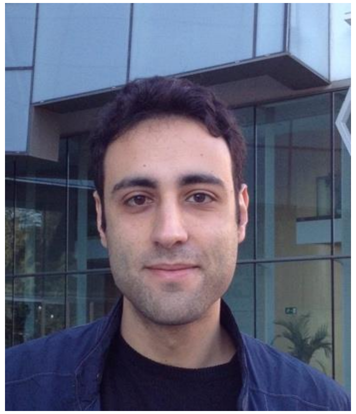
The Power of Graph Sparsification in the Continual Release Model

Quanquan C. Liu
Yale University

Joint work with



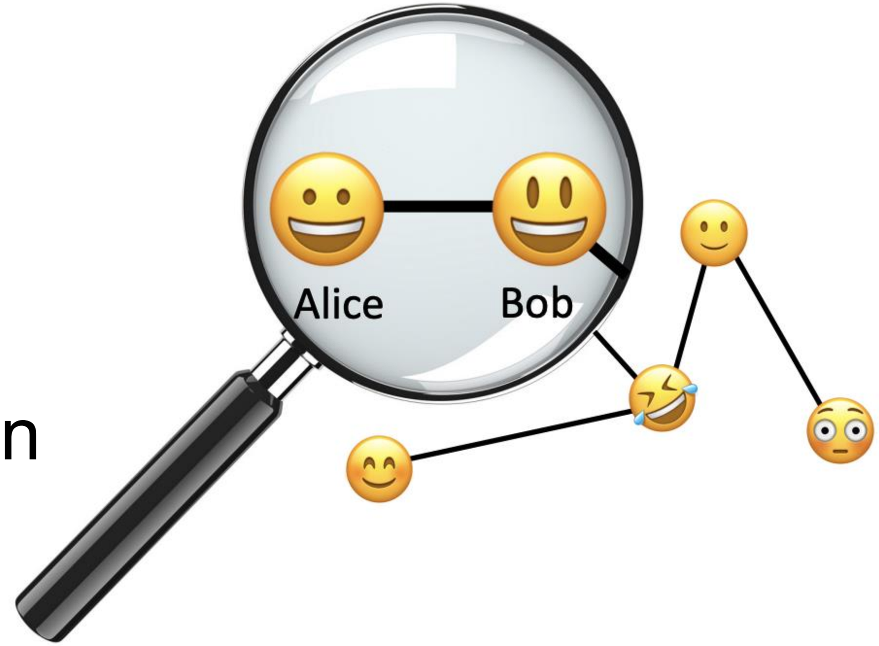
Alessandro Epasto
Google Research

Tamalika Mukherjee
Columbia University

Felix Zhou
Yale University

Publishing Sensitive Graph Information

- Potentially **sensitive connections between individuals** published as graphs
  - Social relationships
  - Financial transactions
  - Disease (e.g. COVID) transmission
  - Search data
  - Email and cell phone communication



Why do we want privacy on graphs?

- Privacy attacks can identify and deanonymize individuals and connections based on **external (e.g. public) information**
  - Re-identify nodes in **social networks** and **computer networks**

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Wherefore art thou r3579x?: anonymized social networks, hidden patterns, and structural steganography

Authors: Lars Backstrom, Cynthia Dwork, and Jon Kleinberg | Authors Info & Claims

WWW '07: Proceedings of the 16th international conference on World Wide Web • May 2007 • Pages 181 - 190

Playing Devil's Advocate:
Inferring Sensitive Information from Anonymized Network Traces

Scott E. Coull* Charles V. Wright* Fabian Monrose* Michael P. Collins† Michael K. Reiter‡

Graph Data Anonymization, De-Anonymization Attacks, and De-Anonymizability Quantification: A Survey

Publisher: IEEE

Shouling Ji ; Prateek Mittal ; Raheem Beyah

A Practical Attack to De-anonymize Social Network Users

Publisher: IEEE

Gilbert Wondracek ; Thorsten Holz ; Engin Kirda ; Christopher Kruegel

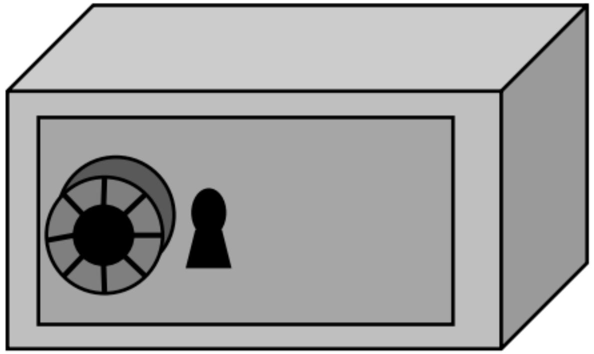
Link Prediction by De-anonymization: How We Won the Kaggle Social Network Challenge

Arvind Narayanan, Elaine Shi, Benjamin I. P. Rubinstein

Private Analysis of Graph Data

Graph G

Trusted Curator



Queries

Answers

Users

Researchers
Government
Business
Malicious Adversaries

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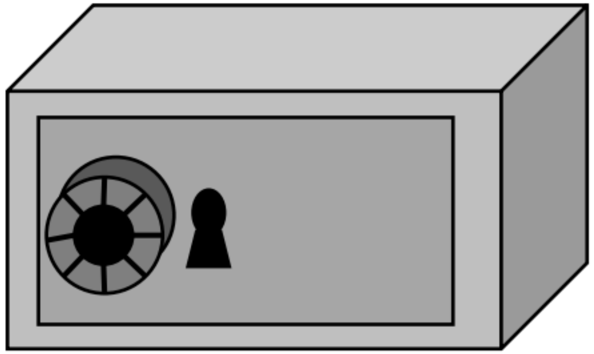

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Answers satisfy **Differential Privacy**

Notation

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- UB = upper bound, LB = lower bound

Differential Privacy

Differential Privacy [Dwork-McSherry-Nissim-Smith '06]

A (randomized) algorithm $\mathcal{A}$ is **ε-differentially private** if for all pairs of neighbors G and G' and all sets of possible outputs Y:

$$e^{-\varepsilon} \leq \frac{\Pr[\mathcal{A}(G) \in Y]}{\Pr[\mathcal{A}(G') \in Y]} \leq e^{\varepsilon}$$

Neighboring Graphs

Edge-neighboring graphs
differ in **1 edge**

G

G'

Neighboring Graphs

Edge-neighboring graphs differ in **1 edge**

Node-neighboring graphs differ in **all edges adjacent to any 1 node**

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- Continuously release **accurate graph statistics** **after each update** while using sublinear space

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  - Edge coloring [Ghosh-Stoeckl ‘23]

Continual Release Model [DNPR10, CSS11]

- Given a stream of T **edge updates** (insertions and deletions), produce a vector of T **outputs**, one after every update

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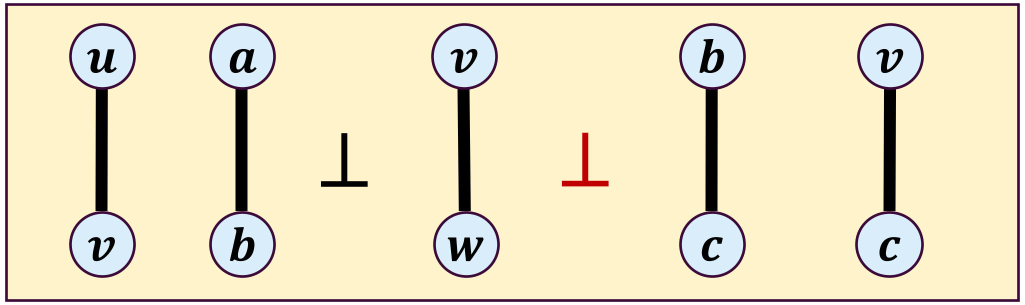
  - Focus on **insertion-only streams**
  - An update is either an **edge insertion** or $\perp$

Neighboring Streams

Edge-neighboring streams differ in **1 edge insertion**



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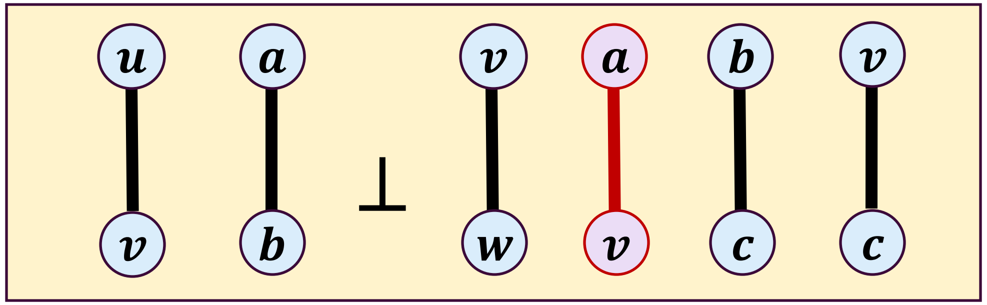

S'

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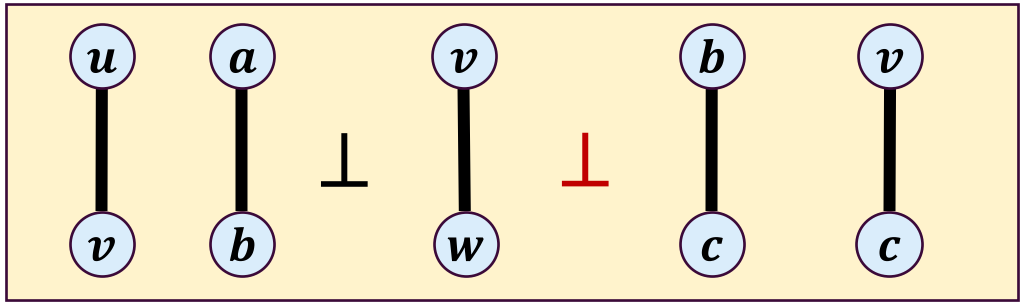

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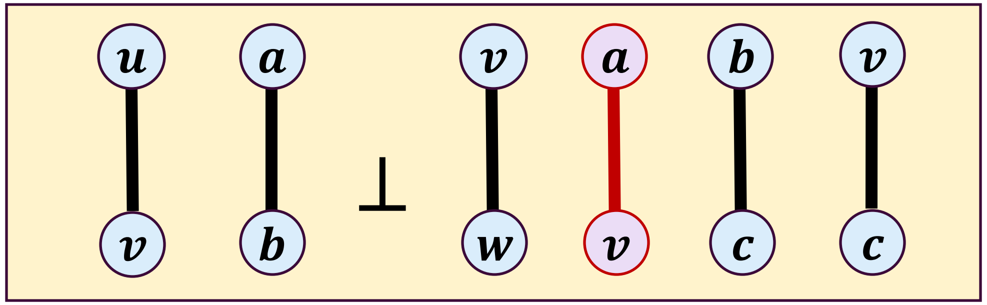

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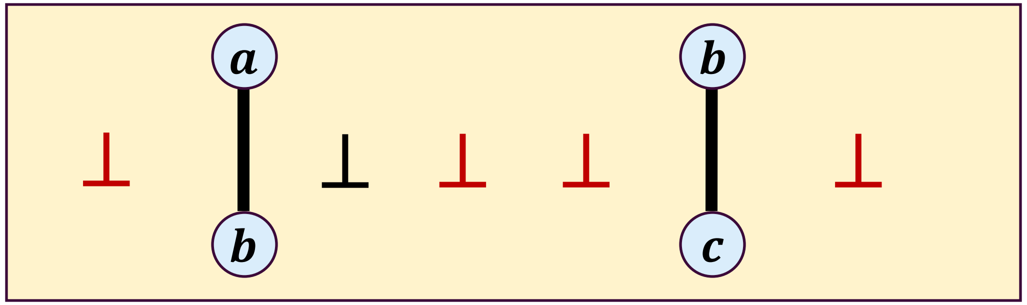
Node-neighboring streams differ in **all edge insertions adjacent to 1 vertex**



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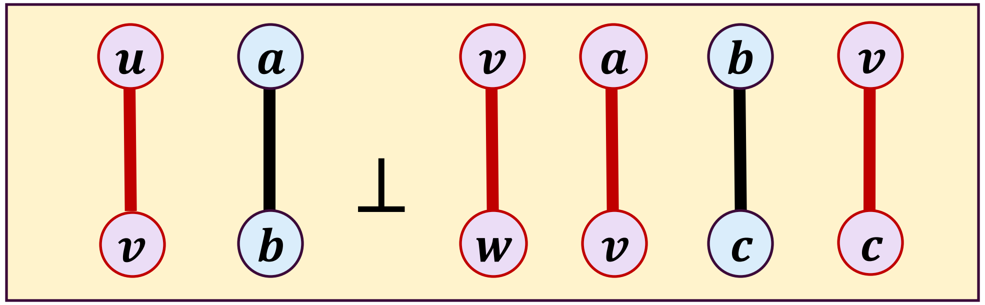

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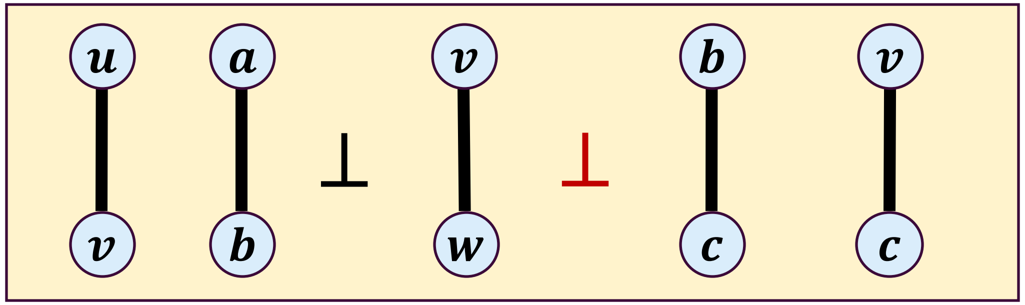

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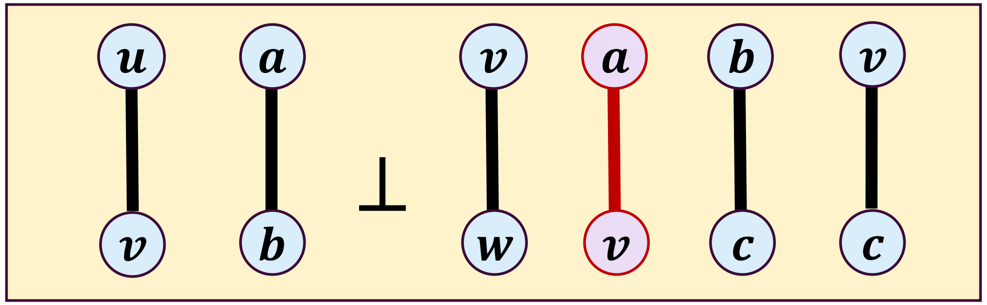

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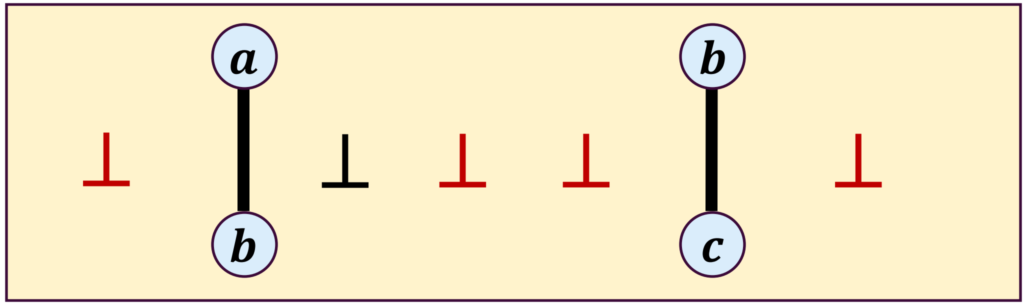

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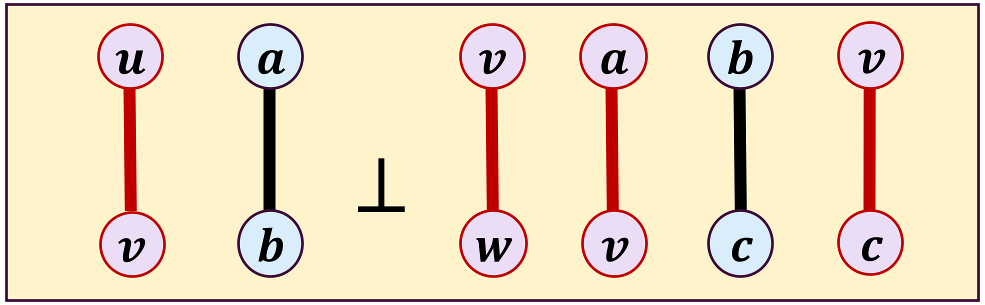

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- In static setting, only **one release**
- Unlike static setting, $\boldsymbol{T}$ **releases in continual release**
- If edge that differs **occurs early in the stream**, **each release loses privacy**
- **Composition** over $\boldsymbol{T}$ **releases** could result in $O\left(\frac{T}{\varepsilon}\right)$ error

Previous Work in Graph Continual Release

- Release **numerical valued solutions** for many graph problems [Song-Little-Mehta-Vinterbo-Chaudhuri ‘18, Fichtenberger-Henzinger-Ost ‘21, Jain-Smith-Wagaman ’24]

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  - Minimum spanning tree size
  - Minimum cut size
  - Maximum matching size
  - Edge count
  - Degree histogram
  - Triangle count
  - k-star count

Previous Work in Graph Continual Release

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  - **Prefix sum** of difference sequence then is **approximate solution**
  - Binary tree mechanism and SVT reduces additive error to $\frac{\text{poly}(\log n)}{\varepsilon}$ [FHO21]

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Previous Work in Graph Continual Release

- DP on **node-neighboring** streams [SLMVC18, FHO21, JSW24]:
  - Requires **bounded degree** graph streams for $\mathrm{poly}(\log n)$ additive error [SLMVC18, FHO21]
  - Or **nearly bounded degree** graph streams where number of nodes with unbounded degree is at most $\mathrm{poly}(\log n)$ [JSW24]

Previous Work in Graph Continual Release

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  - Cannot achieve **sublinear space guarantees** in the number of edges as in non-private streaming algorithms

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- **Caveat 2**: Can return only **value of solution** instead of vertex subsets

- **Caveat 3**: Can return non-trivial node-privacy guarantees for (nearly) **bounded-degree streams**

Our Contributions

Sublinear space continual release graph algorithms

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Returns **vertex subset solutions** in continual release

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Node-private algorithms for **bounded arboricity** graphs in continual release

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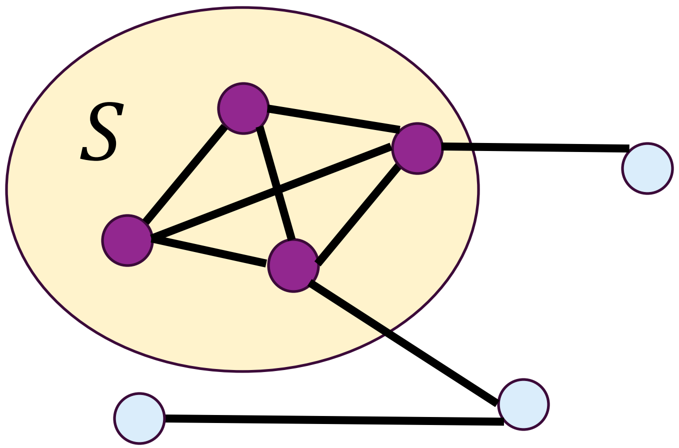
Sublinear space continual release graph algorithms
Returns **vertex subset solutions** in continual release
Node-private algorithms for **bounded arboricity** graphs in continual release

- First continual release algorithm for k-core decomposition

Our Contributions: Densest Subgraph

- Find an induced subgraph $S \subseteq G$ with maximum induced density, $\max_{S \subseteq G} \left(\frac{E(S)}{V(S)} \right)$



Densest subgraph is S with density $\frac{3}{2}$

Our Contributions: Densest Subgraph

Edge-DP

Our Results

- **Vertex Subset**
- $\tilde{O}\left(\frac{n}{\varepsilon}\right)$ **space**
- **UB:** $\left(1+\eta, \frac{\log^5 n}{\varepsilon}\right)$

Continual Release

[FHO21, JSW24]

- Density value-only
- $\Theta(m)$ space
- **UB**: $\left(1+\eta, \frac{\log^2 n}{\varepsilon}\right)$

Non-Private

[MTVV15, EHW16]

- $(1+\eta, 0)$, $\tilde{O}(n)$

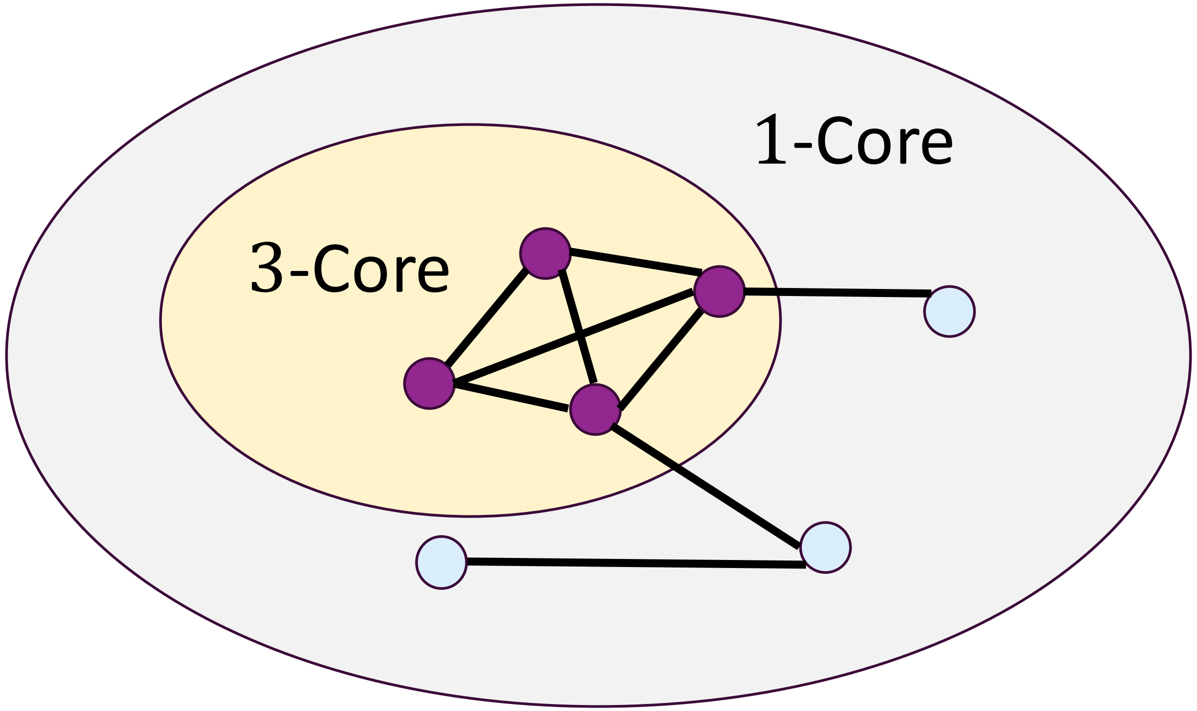
Static

[DLRSS22, DLL23, DKLV24]

- Vertex Subset
- **UB**: $\left(1+\eta, \frac{\log^4 n}{\varepsilon}\right)$
- **LB**: $\left(\beta, \Omega\left(\frac{1}{\beta}\sqrt{\frac{\log(n)}{\varepsilon}}\right)\right)$

Our Contributions: k-Core Decomposition

- Decomposition of nodes of G into cores where each k-core is a maximal induced subgraph with induced degree at least k



Graph contains a 1-core and 3-core

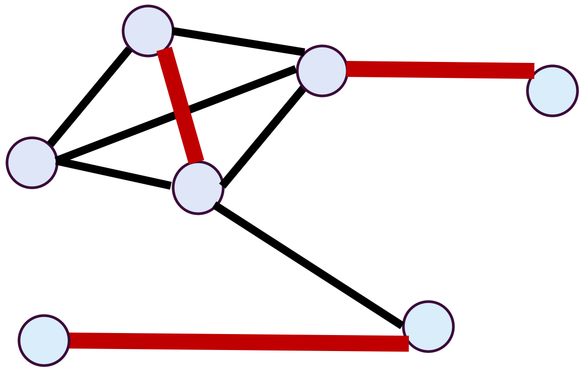
Our Contributions: k-Core Decomposition

Edge-DP

Our Results	Continual Release	Static [DLL23, HSZ24]
• $\tilde{O}\left(\frac{n}{\varepsilon}\right)$ space • UB: $\left(2+\eta, \frac{\log^3 n}{\varepsilon}\right)$	• None **Non-Private** [Esfandiari-Lattanzi-Mirrokni '18] • $(1+\eta, 0)$, $\tilde{O}(n)$ space	• **UB**: $\left(1, O\left(\frac{\log(n)}{\varepsilon}\right)\right)$ • **LB**: $\left(\beta, \Omega\left(\frac{\log(n)}{\varepsilon\beta}\right)\right)$

Our Contributions: Maximum Matching Size

- Find a matching (pairing of nodes where no node is paired with more than one other node) of maximum size

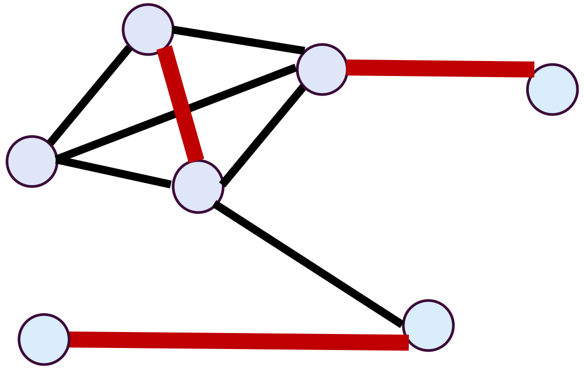


Maximum matching size: 3

Our Contributions: Maximum Matching Size

- Find a matching (pairing of nodes where no node is paired with more than one other node) of maximum size

Cannot differentially privately release set of edges in the matching



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Our Contributions: Maximum Matching Size

Edge-DP

Our Results

- $O\left(\frac{\text{poly}(\log n)}{\varepsilon}\right)$ **space**
- **UB:** $\left((1+\eta)(2+\tilde{\alpha}), \frac{\log^3 n}{\varepsilon}\right)$

Continual Release

[FHO21, JSW24]

- $\Theta(m)$ space
- **UB**: $\left(1+\eta, \frac{\log^2 n}{\varepsilon}\right)$
- **LB**: $\left(1, \Omega(\log n)\right)$

Non-Private

[McGregor-Voronikova ‘18]

- $(1+\eta)(2+\tilde{\alpha}), O(\log n)$

Our Contributions: Maximum Matching Size

Edge-DP

Our Results

- $O\left(\frac{\log^3 n}{\varepsilon}\right)$ **space**
- **UB:** $\left((1+\eta)(2+\tilde{\alpha}), \frac{\log^3 n}{\varepsilon}\right)$

$\tilde{\alpha}$ is a public bound on the arboricity

Continual Release

[FHO21, JSW24]

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Our Contributions: Maximum Matching Size

Node-DP

Our Results

- $O(n\widetilde{\alpha})$ **space**
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Continual Release

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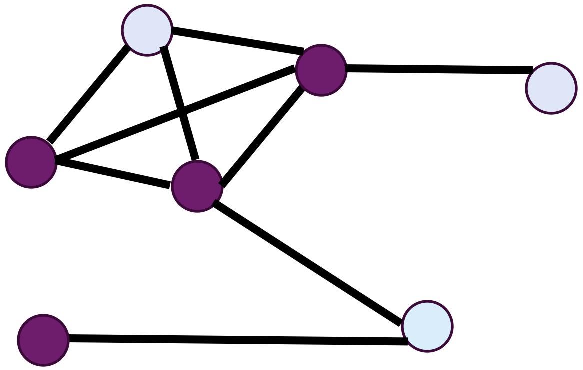
Our Contributions: Implicit Vertex Cover

- Find a minimum sized set of vertices where every edge has at least one endpoint in the set



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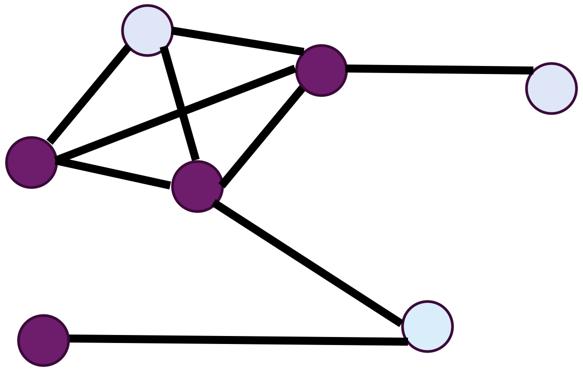
Implicit Vertex Cover releases information such that every edge knows which vertex covers it



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Node-DP

Our Results (One Shot)	Continual Release	Static [GLMRT10]

• $O(n\tilde{\alpha})$ space
• **UB:** $\left(3+\eta + O\left(\frac{\tilde{\alpha}}{\varepsilon}\right), O\left(\frac{\tilde{\alpha}\log n}{\varepsilon}\right)\right)$
	• None	• **None** for Node-DP
		• Edge-DP:
		  • **UB:** $\left(2+\frac{16}{\varepsilon}, 0\right)$
		  • **LB:** $\left(\Omega\left(\frac{1}{\varepsilon}\right), 0\right)$

Fully Dynamic Lower Bounds

Edge-DP

Our Results

- **Matching size, triangle count, connected components**
- **LB:** $\left(\mathbf{1}, \min\left(\sqrt{\frac{n}{\varepsilon}}, \frac{T^{1/4}}{\varepsilon^{3/4}}\right)\right)$

Continual Release

[FHO21]

- Matching size, triangle count
- **LB**: $(1, \Omega(\log T))$

Main Technique: Graph Sparsification

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- Determine property of the sparsified graph as an **approximate answer** of the original graph

- Sparsification can be deterministic or randomized

  - Randomized approaches include various **edge sampling algorithms**

Main Technique: Graph Sparsification

- Previous work used sparsification in DP
  - Static DP setting [Upadhyay ’13, Arora-Upadhyay ‘19]
  - Sliding window DP model [Arora-Upadhyay-Upadhyay ‘21]

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    - Stream satisfying degree bound $\widetilde{D}$
    - Identical to every prefix of stream with vertices is $\widetilde{D}$-bounded

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  - Results in graph with degree upper bounded by $\widetilde{D}$

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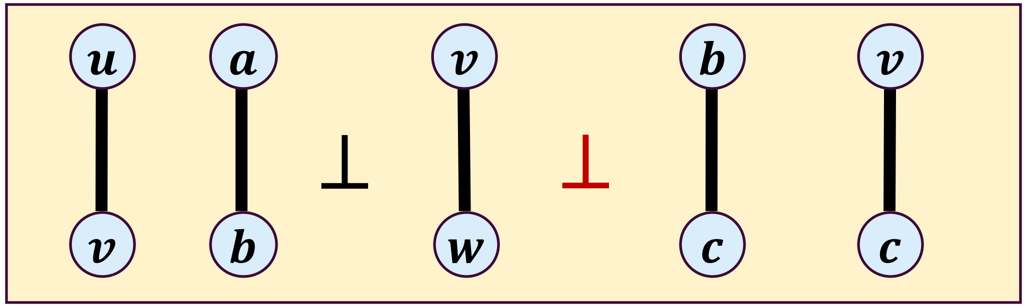
Edge-neighboring with $\widetilde{D} = 3$



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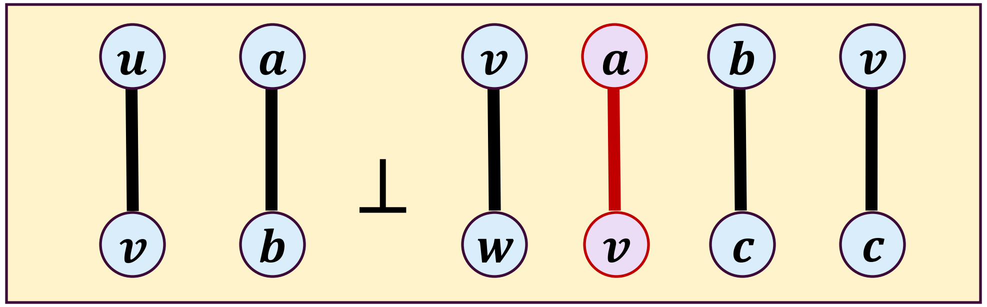
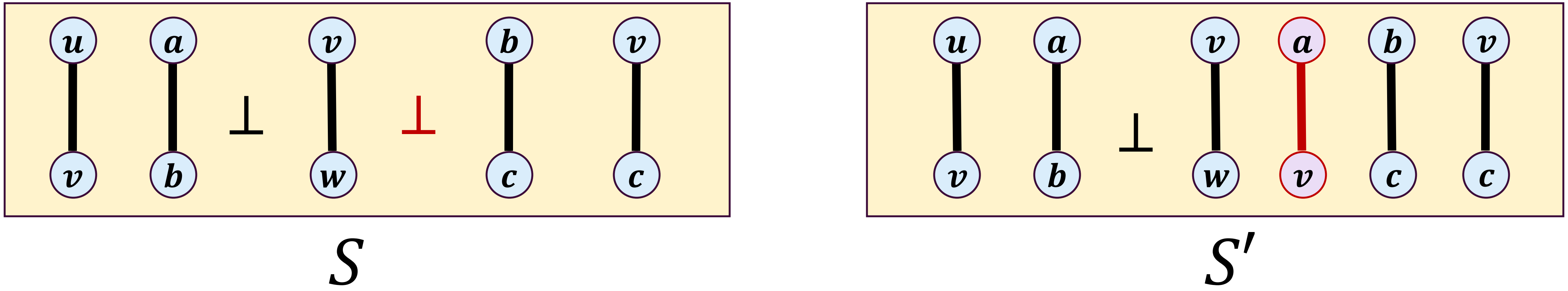


S'

Challenges of Sparsification in Continual Release

Edge-neighboring with $\widetilde{D} = 3$



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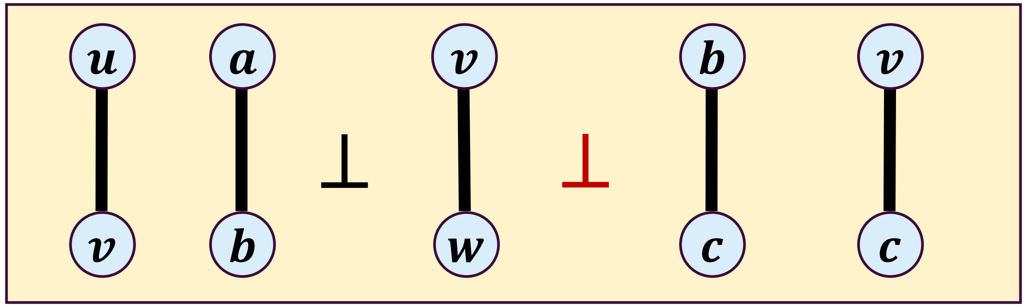

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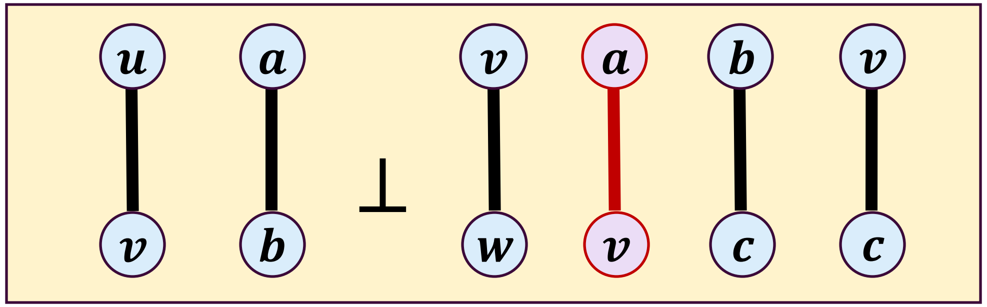

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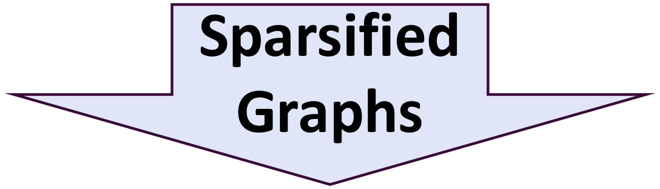
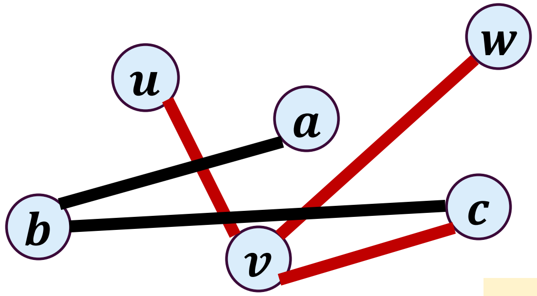
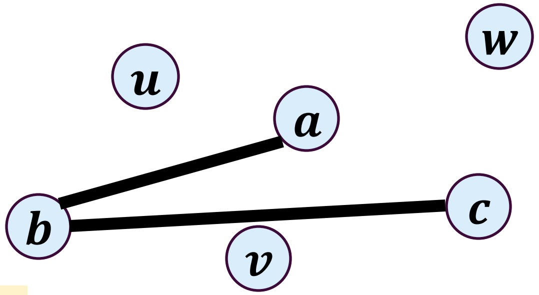




G

Differs by $\widetilde{D}$ edges!



G'

Sparsification Need to Preserve Edge Edit Distance

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  - **Randomized** sparsification algorithms
    - Exists coupling of randomness where output streams differ by bounded number of events

Sublinear Space Densest Subgraph

- **Our results**:
  - Vertex Subset
  - $\tilde{O}\left(\frac{n}{\varepsilon}\right)$ space
  - UB: $\left(1+\eta, \frac{\text{poly}(\log n)}{\varepsilon}\right)$-approximation

Sublinear Space Densest Subgraph

- Uses uniform sampling idea of [McGregor-Tench-Vorotnikova-Vu ‘15] and [Esfandiari-Hajiaghayi-Woodruff ‘16]

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  - Find densest subgraph in sample, return vertex set as densest subgraph in original, scale by $1/p$ for size

Sublinear Space Densest Subgraph

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    - Ensure adaptive sampling probability is **edge edit distance preserving**

Sublinear Space Densest Subgraph

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Compounding Errors in Continual Release

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Sublinear Space Densest Subgraph

- Finally, additional challenge:
  - Additive noise of $O\left(\frac{\log n}{\varepsilon}\right)$

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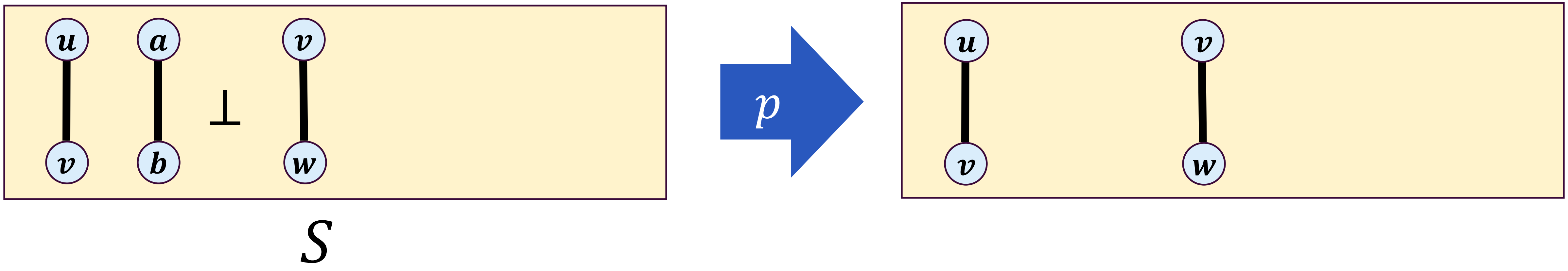
  - **Solution**: Ensure returned densest subgraph has large enough size

Sublinear Space Densest Subgraph

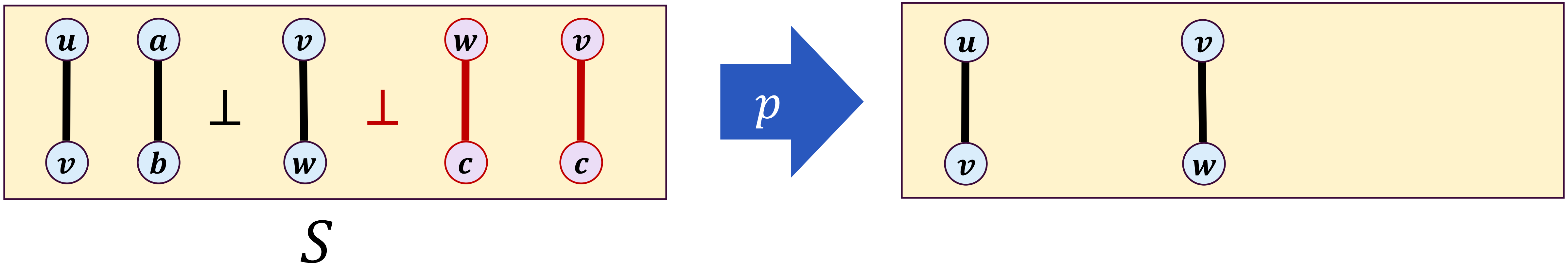
Putting it Together



Sample each edge update with probability $p = \Theta\left(\frac{n \log n}{m'}\right)$

Sublinear Space Densest Subgraph

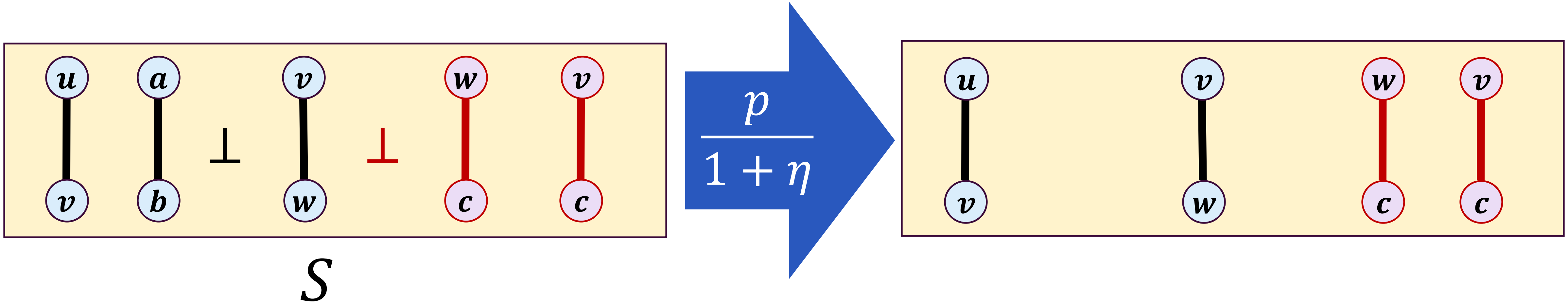
Putting it Together



Use SVT to determine when threshold of number of edges seen exceeds $(1 + \eta)m'$

Sublinear Space Densest Subgraph

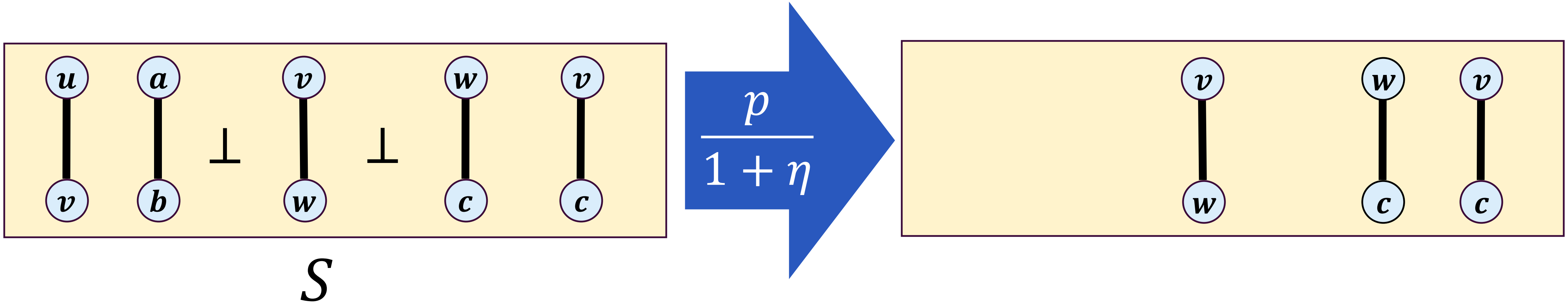
Putting it Together



If SVT is satisfied decrease probability by $(1+\eta)$ factor

Sublinear Space Densest Subgraph

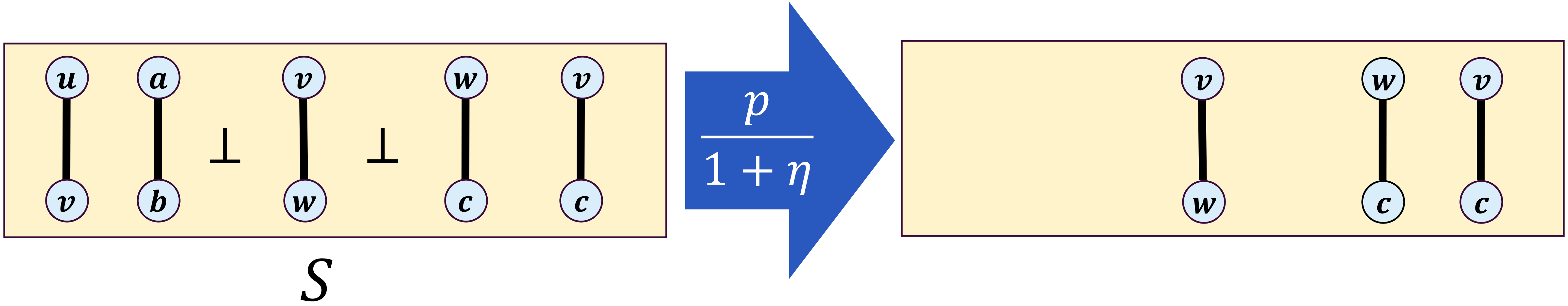
Putting it Together



Resample existing sampled edges with probability $\frac{1}{1+\eta}$

Sublinear Space Densest Subgraph

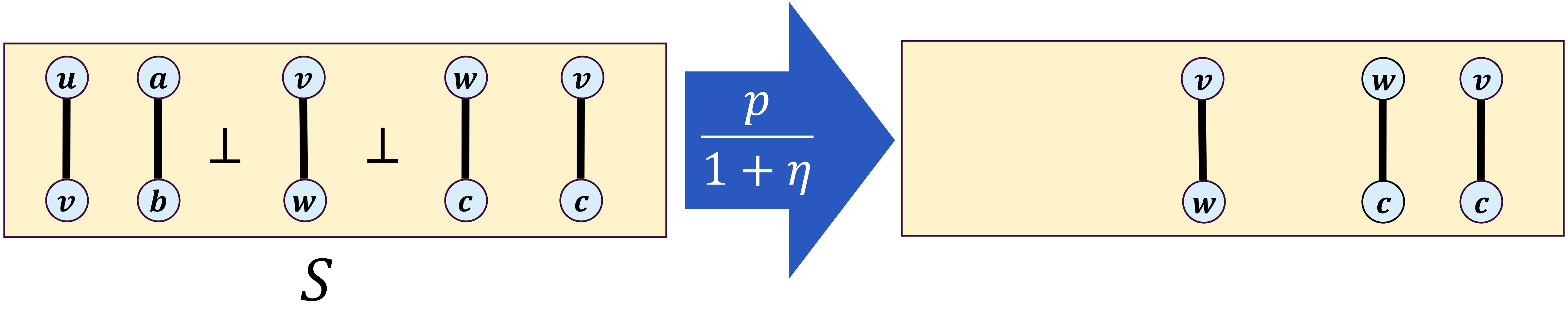
Putting it Together



Use ε-DP algorithm for each sample to determine released solution at **appropriate timestamps**

Sublinear Space Densest Subgraph

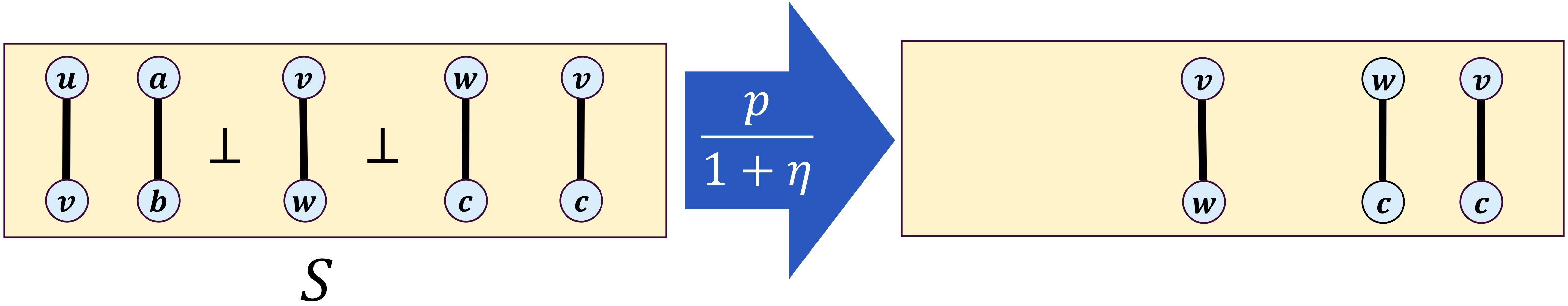
Putting it Together



Use SVT to determine if densest subgraph **increased in value by $(1+\eta)$-factor**

Sublinear Space Densest Subgraph

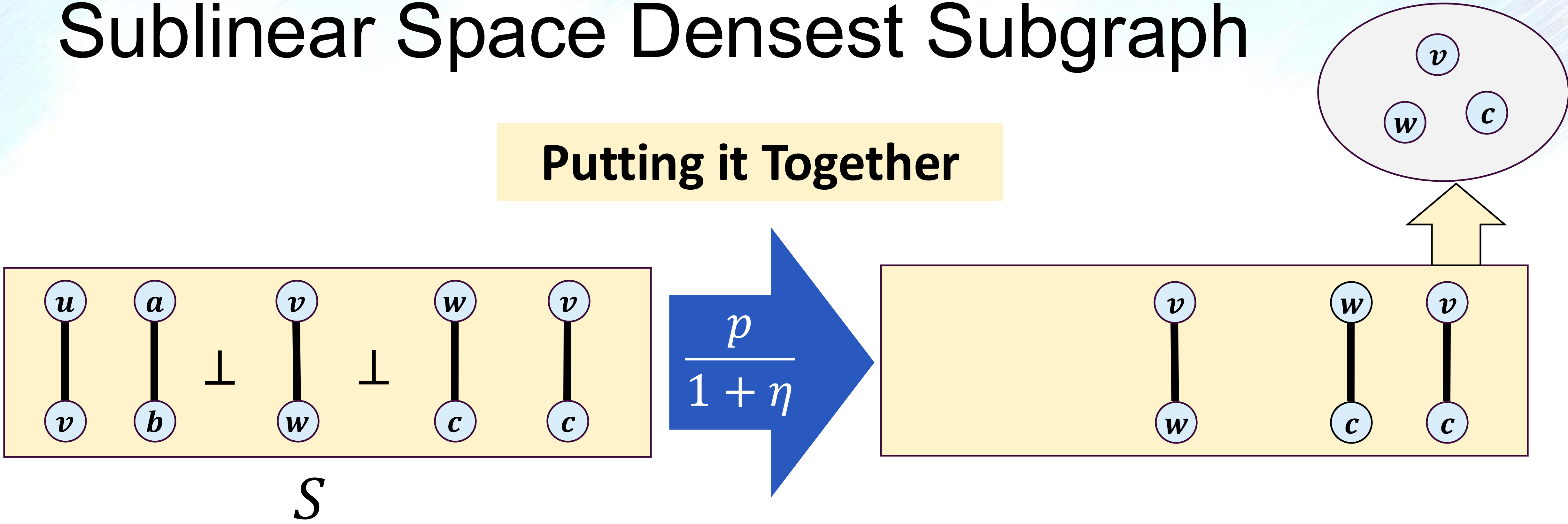
Putting it Together



Only release when densest subgraph **increased in value by $(1+\eta)$-factor**

Sublinear Space Densest Subgraph

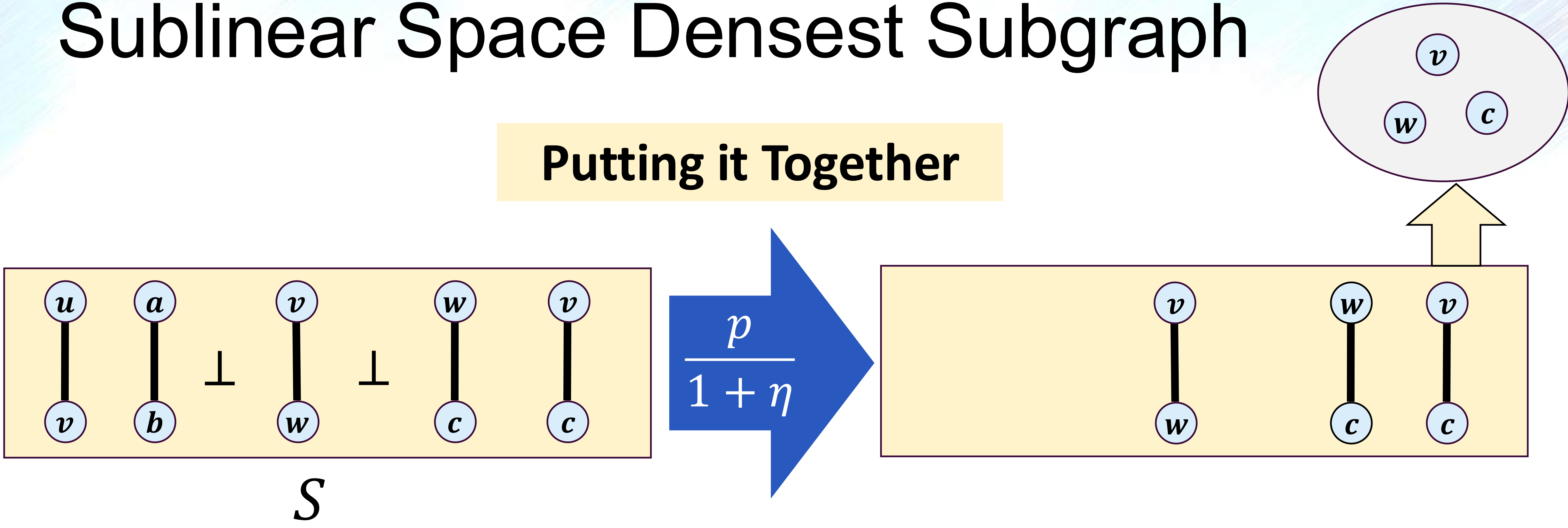
Putting it Together



Only release subset of vertices when densest subgraph **increased in value by $(1+\eta)$-factor**

Sublinear Space Densest Subgraph

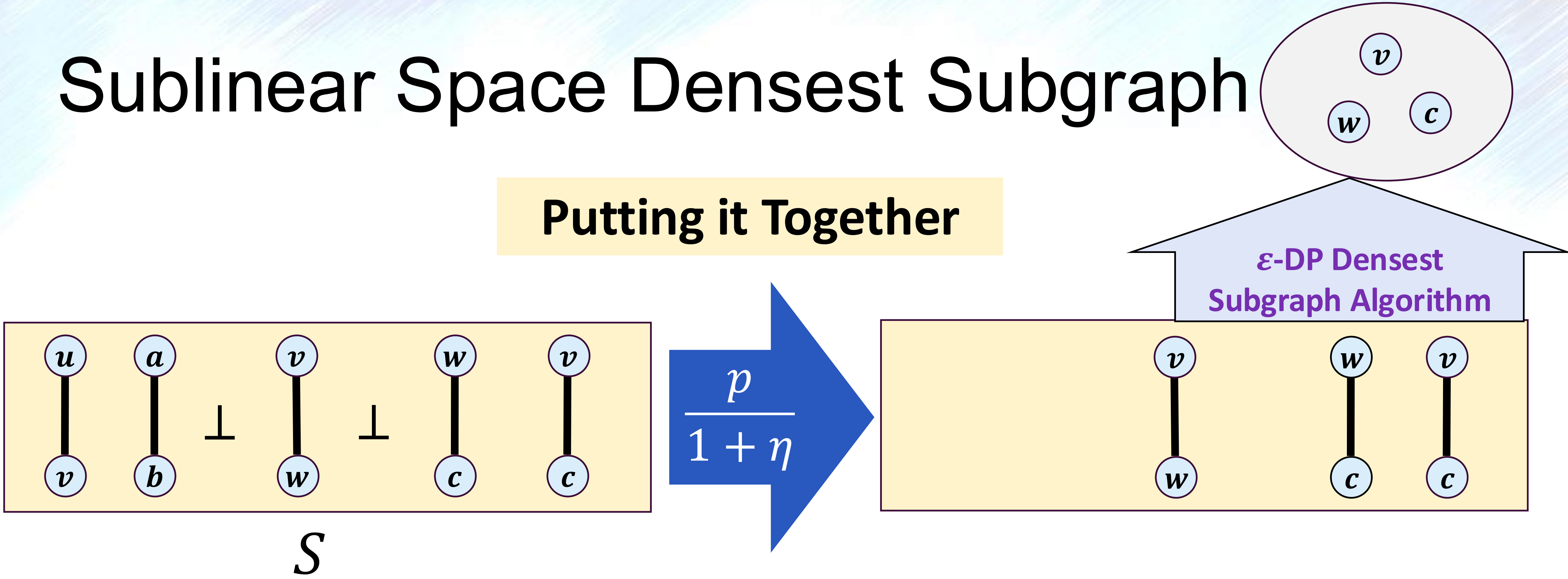
Putting it Together



Scale value of densest subgraph by **inverse of current sampling probability**

Sublinear Space Densest Subgraph

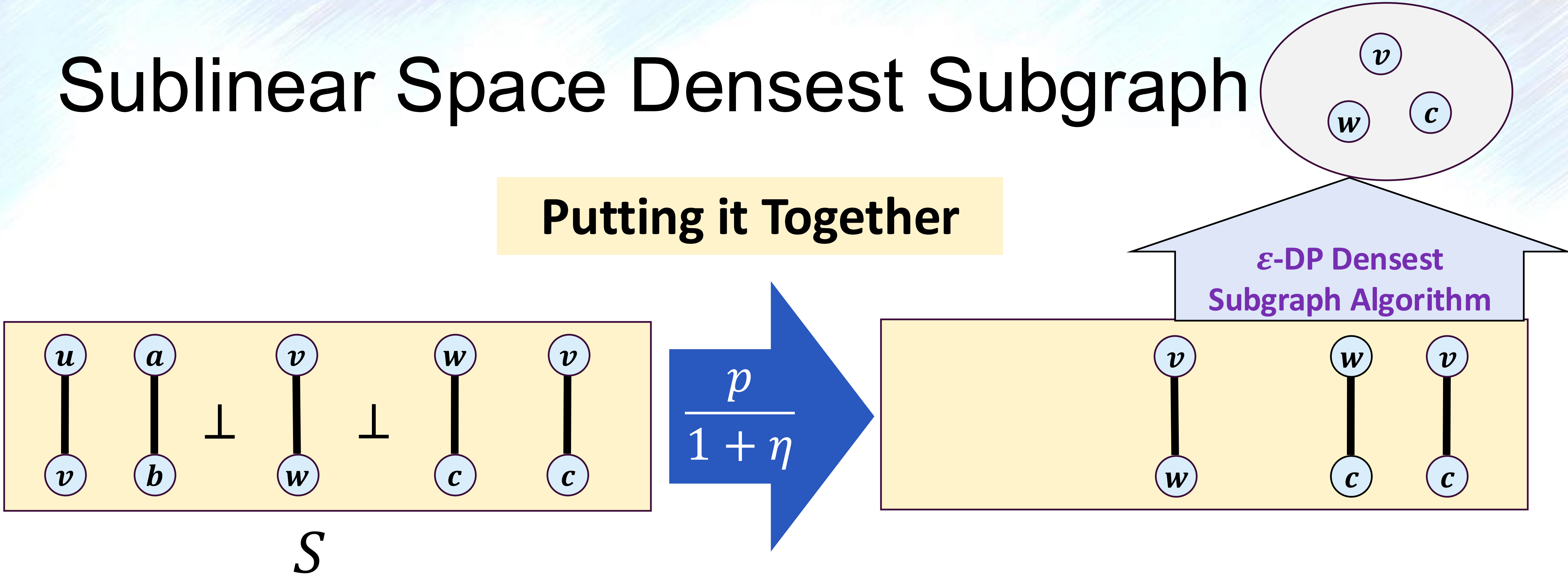
Putting it Together



Use any ε-DP (static) Densest Subgraph algorithm for determining subset of vertices to release

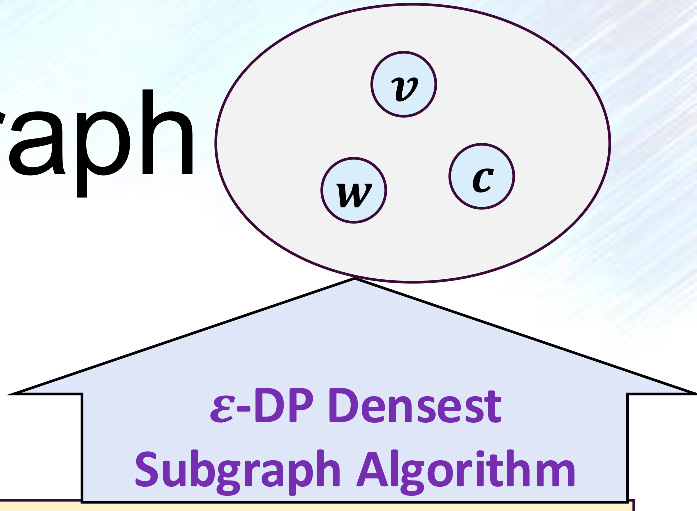
Sublinear Space Densest Subgraph

Putting it Together



Set first non-trivial initial release for density to be $\Omega\left(\frac{\log^2 n}{\varepsilon}\right)$ to account for DP alg additive error

Sublinear Space Densest Subgraph



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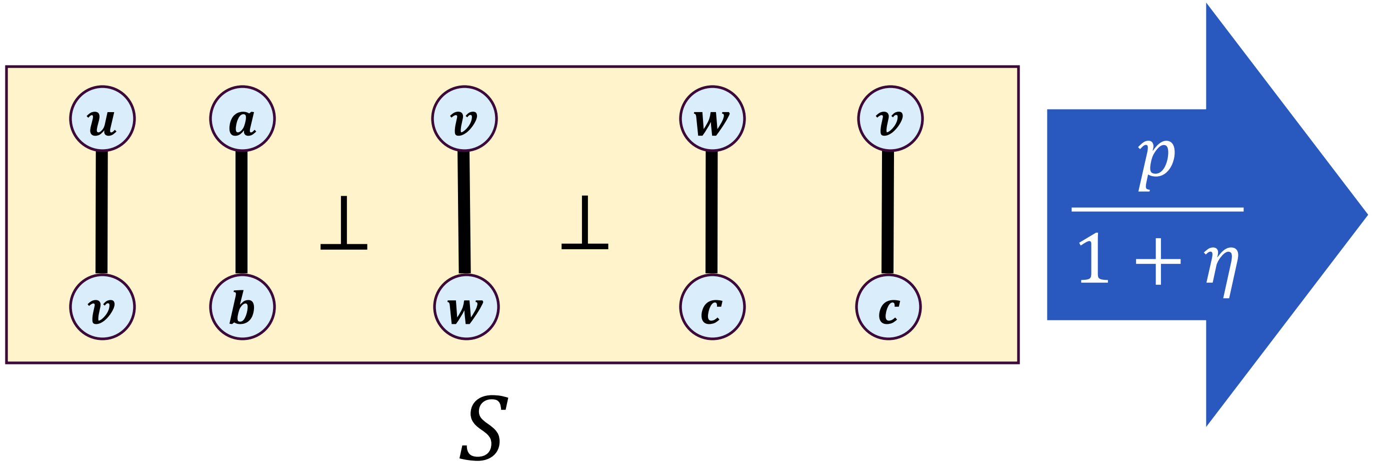
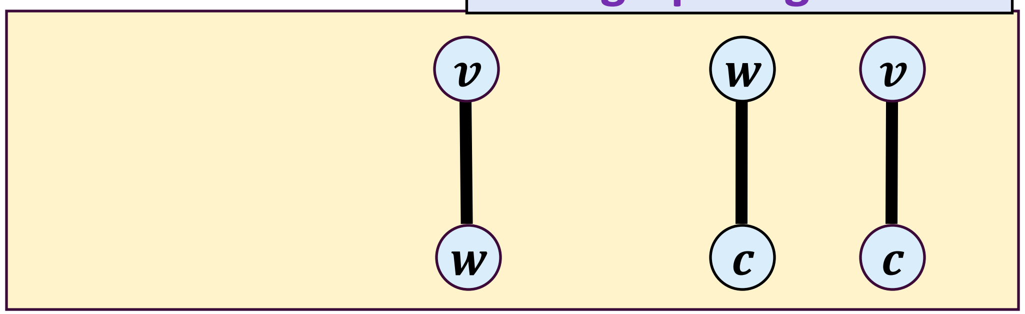
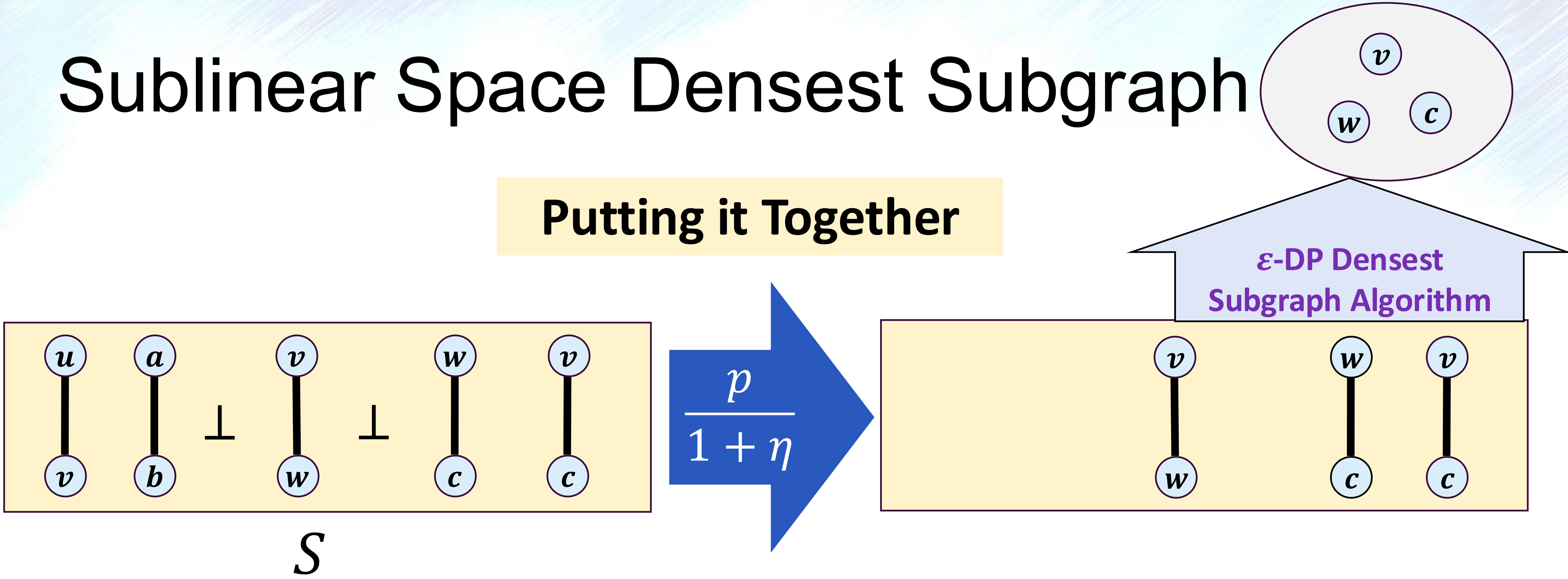




ε-DP Guarantee from DP of SVT, Edge Edit Distance is preserved, and Composition

Sublinear Space Densest Subgraph

Putting it Together



Approximation guarantee from very intricate **Chernoff Bound argument accounting for errors from SVT and DP algorithm**

Sublinear Space Densest Subgraph

ε-DP Densest
graph Algorithm

One Takeaway: adaptive uniform sampling with SVT is a sublinear simplification in DP that is edge distance preserving

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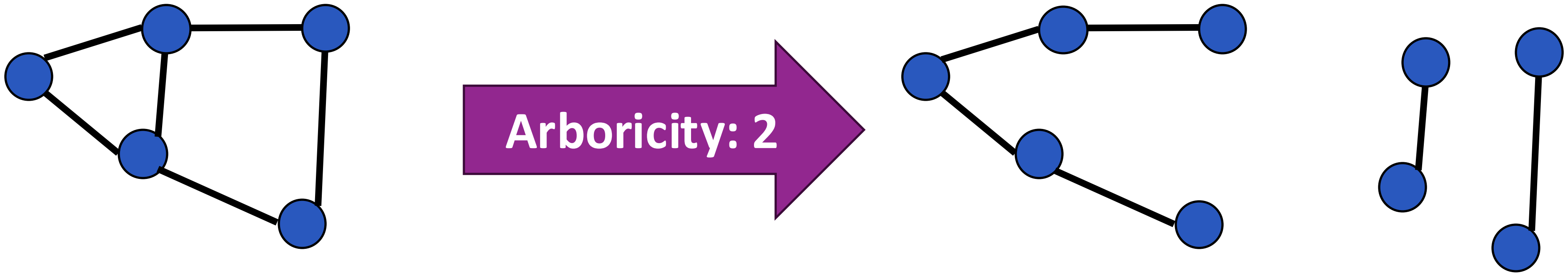
Node-DP Maximum Matching

- **Arboricity sparsification**: sparsification using upper bound based on the arboricity α
  - **Arboricity:** minimum number of forests to decompose a graph
    - Measure of local sparsity



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  - Matching in sparsified graph is a $(1+\eta)$-approximation of the maximum matching in the original graph

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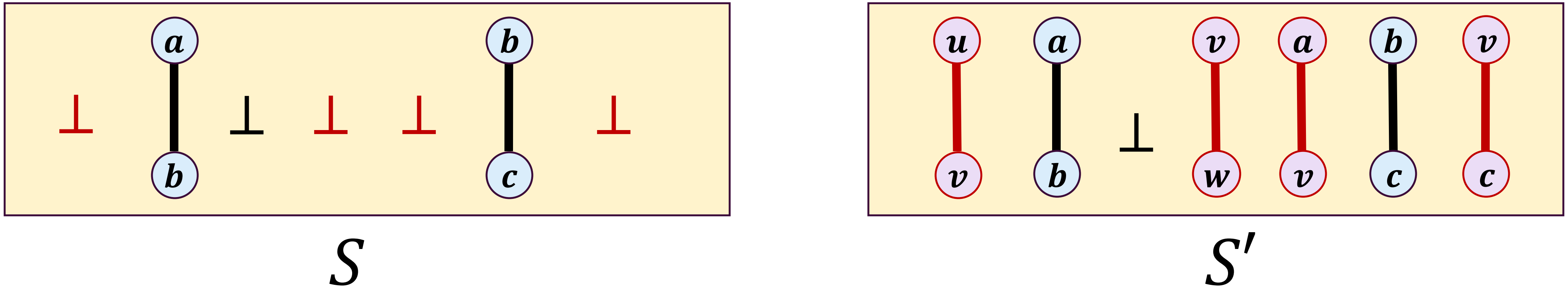
Node-neighboring streams and given $\tilde{\alpha} = 2$



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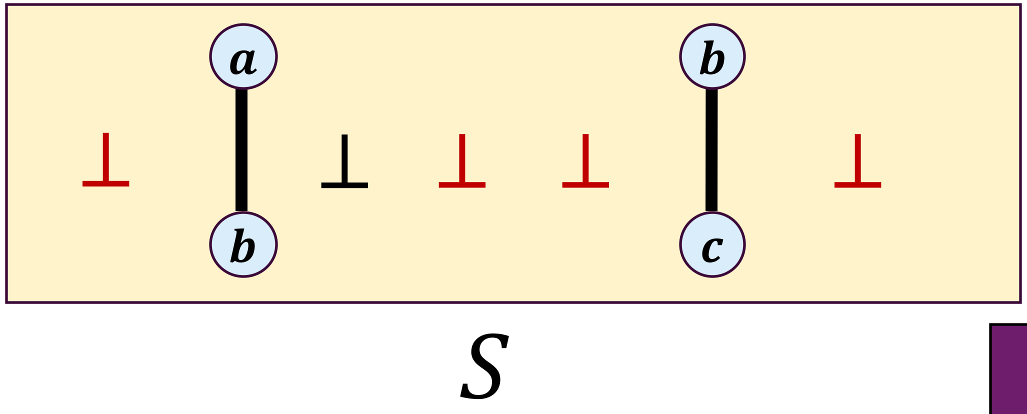
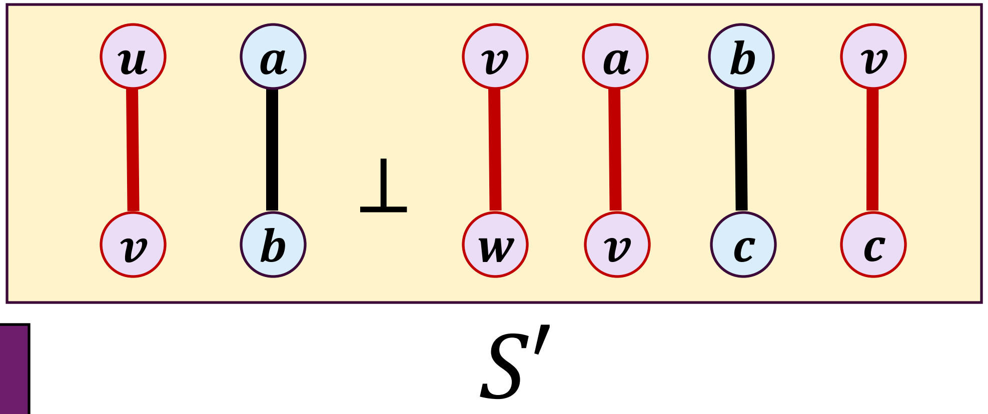
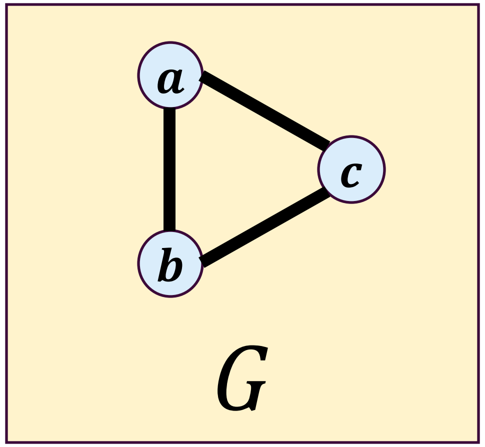
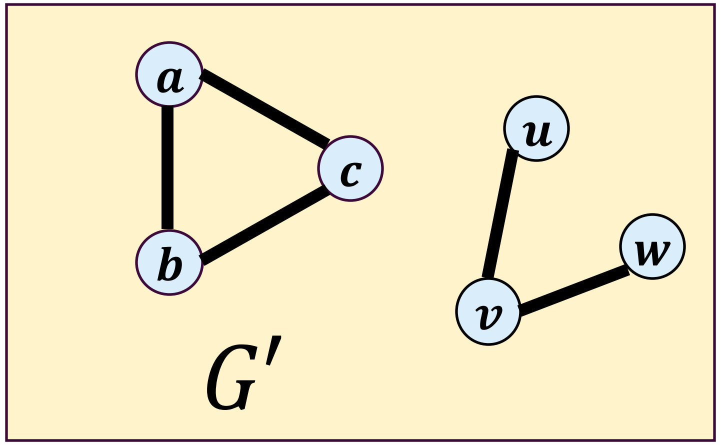






Edge edit distance is $\tilde{\alpha}$



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One Takeaway: arboricity sparsification for node-DP applies to vertex cover and DP in the static setting

Open Questions

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  - Is this factor necessary?